

A quantum protocol to win the graph colouring game on all Hadamard graphs

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Abstract : This paper deals with graph colouring games, an example of pseudo-telepathy, in which two provers can convince a verifier that a graph G is c -colourable where c is less than the chromatic number of the graph. They win the game if they convince the verifier. It is known that the players cannot win if they share only classical information, but they can win in some cases by sharing entanglement. The smallest known graph where the players win in the quantum setting, but not in the classical setting, was found by Galliard, Tapp and Wolf and has 32,768 vertices. It is a connected component of the Hadamard graph G_N with $N = c = 16$. Their protocol applies only to Hadamard graphs where N is a power of 2. We propose a protocol that applies to all Hadamard graphs. Combined with a result of Frankl, this shows that the players can win on any induced subgraph of G_{12} having 1609 vertices, with $c = 12$. Combined with a result of Frankl and Rodl, our result shows that all sufficiently large Hadamard graphs yield pseudo-telepathy games.

1 Introduction

It is known that quantum entanglement allows for a phenomenon called pseudo-telepathy, that is, two parties pretend to be endowed with telepathic powers, as described in a survey paper by Brassard, Broadbent and Tapp [1]. For two parties A and B , a pair of question $(q_a, q_b) \in Q_A \times Q_B$ are given and then a pair of answer $(a_a, a_b) \in A_A \times A_B$ are returned. In an initial phase, the parties can communicate with each other and share information. If the parties share an entangled state, then this setting is called shared entanglement. In the second phase, the parties are no longer allowed to communication with each other before revealing their answers. The parties *win* this instance, if $a_a = a_b \Leftrightarrow q_a = q_a$. A protocol is successful with probability p if it wins any instance that satisfies the promise with probability at least p . This exchange is called a *pseudo-telepathy game* if there is a protocol that is successful with probability 1 with shared entanglement and does not admit such a protocol that is successful with probability 1 without sharing entanglement.

The graph colouring game is an example of a pseudo-telepathy game (Brassard, Cleve and Tapp [1], Cleve, Høyer, Toner and Watrous [3]). In a *graph colouring game*, there are two provers, called Alice and Bob, and a verifier. A graph $G(V, E)$ and an integer c is given to Alice and Bob. Alice and Bob agree on a protocol to convince the verifier that G is c -colourable. The verifier sends $a \in V$ to Alice and sends $b \in V$ to Bob such that $a = b$ or $(a, b) \in E$. Alice and Bob are not permitted to communicate after receiving a and b . Alice sends the colour c_A

of a to the verifier and Bob sends the colour c_B of b to the verifier. Alice and Bob win if $a \neq b$ and $c_A \neq c_B$ or $a = b$ and $c_A = c_B$, and lose otherwise. They win the game if they convince the verifier.

The chromatic number $\chi(G)$ of a graph G is the smallest number of colours that can be assigned to vertices such that no two adjacent vertices receive the same colour. If $c \geq \chi(G)$ then there exists a protocol to win with probability 1 by using a colouring of G with c colours. Otherwise, Alice and Bob cannot win the game with probability 1 using classical methods [2]. Using shared entanglement however, there are graphs where they can win in this situation. The Hadamard graphs, defined by Ito [9, 10], provide such examples.

Let $N = 4k$ for any positive integer k . The *Hadamard graph* G_N is defined as the graph whose vertex set $V_N = \{0, 1\}^N$ and edge set $E_N = \{(u, v) \in V_N^2 \mid d_H(u, v) = N/2\}$, where $d_H(u, v)$ means Hamming distance of u and v . Hadamard graphs are related to Hadamard matrices, which are an important object of study in combinatorics, especially design theory (e.g., see Stinson [12]). One of the open problems is to decide whether for every k , there exists a Hadamard matrix of order $4k$.

With shared entanglement Brassard, Cleve and Tapp [2] showed that Alice and Bob win the graph colouring game with probability 1 for G_{2^n} and $c = 2^n$ by using the Deutsch-Jozsa protocol and by sharing an entangled state. A result of Frankl and Rodl [5] (Theorem 1.11) implies that for all large enough n , $\chi(G_{2^n}) > 2^n$, and so asymptotically the game is an example of a pseudo-telepathy game. Galliard, Tapp and Wolf [6] showed that this was already the case for $n = 16$ using a rather complicated combinatorial argument. Thus the graph colouring game for G_{16} and $c = 16$ is a pseudo-telepathy game.

We extend these results in this paper to all Hadamard graphs. In the next section we state known results on the chromatic number of these graphs. In Section 3, we design a protocol to win the graph colouring game for all Hadamard graph with probability 1. Combining these results it is shown that the graph colouring game for G_{12} with $c = 12$ is a pseudo-telepathy game. Furthermore this is the smallest value of N for which G_N is a pseudo-telepathy game with $c = N$, and this holds for any induced subgraph with at least 1069 vertices.. The concluding section proposes a definition of quantum chromatic number and gives some open problems.

2 Chromatic number of Hadamard graphs

It is easily seen that $\chi(G_4) = 4$ and Ito [10] proved $\chi(G_8) = 8$. Given a graph $G = (V, E)$, the *independence number* of the graph, denoted $\alpha(G)$, is the cardinality of the largest subset of vertices such that no two of them are joined by an edge. Let $p \geq 3$ be an odd prime, $q \geq 1$, and $k = p^q$. Frankl [4] showed that

$$\alpha(G_{4k}) = 4 \sum_{i=0}^{k-1} \binom{4k-1}{i} < \frac{4^{4k}}{3^{3k}}.$$

When $p = 3$ and $q = 1$ we get $\alpha(G_{12}) = 268$. An elementary result of graph theory is that $\chi(G) \geq |V|/\alpha(G)$. Therefore $\chi(G_{12}) \geq 4096/268 > 12$. In fact G_N consists of two identical connected components, each having independence number half of that of G_N . Let H be any induced subgraph, having 1609 vertices, of one of the connected components of G_{12} . (In an induced subgraph, two vertices are adjacent if and only if they are adjacent in the original graph.) Then $\chi(H) \geq 1609/134 > 12$.

Since $\chi(G_4) = 4$, G_4 is not a pseudo-telepathy game with $c = 4$. Similarly, G_8 is not a pseudo-telepathy game with $c = 8$. However we will see that H is a pseudo-telepathy game with $c = 12$.

3 A protocol with probability 1 using QFT

In this section we extend the protocol by Brassard, Cleve and Tapp [2]. Their protocol employs the quantum Hadamard transform while our protocol employs the quantum Fourier transform (QFT) with any order, which can be exactly done as shown by Mosca and Zalka [11].

We describe a protocol such that Alice and Bob win the graph colouring game of G_N with probability 1 where $2^{n-1} < N \leq 2^n$. In this protocol, we use the following two operations QFT_N and P_i : QFT_N is a *general* quantum Fourier transform with order N , not necessarily 2^n , defined as

$$\text{QFT}_N : |i\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} (\omega)^{i \cdot j} |j\rangle, \quad (1)$$

where $\omega = \exp\left(\frac{2\pi\sqrt{-1}}{N}\right)$. Mosca and Zalka [11] show that this QFT with any order can be performed exactly. The operation P_i is a phase shift corresponding to the i -th bit of an input string l

$$P_i : |i\rangle \mapsto (-1)^{l_i} |i\rangle. \quad (2)$$

Our protocol has four steps. Alice and Bob can communicate with each other at only step 1.

Step 1: Prepare initial state $|\Psi_{AB}\rangle$

In step I, Alice and Bob prepare $2n$ -qubits $|0\rangle^{\otimes n} \otimes |0\rangle^{\otimes n}$. Alice has the first n -qubits and Bob has the second. Alice first applies QFT_N to her N -qubits:

$$|0\rangle^{\otimes n} \otimes |0\rangle^{\otimes n} \xrightarrow{\text{QFT}_N} \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle \otimes |0\rangle. \quad (3)$$

Bob then applies controlled-NOT operations to his n -qubits with Alice's qubits as control qubits for sharing the initial entanglement states:

$$\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle \otimes |i\rangle =: |\Psi_{AB}\rangle. \quad (4)$$

Step 2: Apply phaseshift P_{a_i} and P_{b_i}

Alice and Bob compute

$$(P_{a_i} \otimes P_{b_i}) |\Psi_{AB}\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} (-1)^{a_i \oplus b_i} |i\rangle \otimes |i\rangle. \quad (5)$$

Step 3: Apply the general quantum Fourier transform

Alice and Bob compute

$$\begin{aligned} & (\text{QFT}_N \otimes \text{QFT}_N^{-1}) (P_{a_i} \otimes P_{b_i}) |\Psi_{AB}\rangle \\ &= \left(\frac{1}{\sqrt{N}}\right)^2 \sum_{j_A=0}^{N-1} \sum_{j_B=0}^{N-1} \sum_{i=0}^{N-1} (\omega)^{i \cdot (j_A - j_B)} (-1)^{a_i \oplus b_i} |j_A\rangle \otimes |j_B\rangle. \end{aligned} \quad (6)$$

Step 4: Measure

Alice and Bob measure her(his) qubits using the computational bases and each obtain one of N basis states. They output colours corresponding to their measurement result to verifier.

We have the following theorem for our protocol.

Theorem 3.1 *Alice and Bob win the game with probability 1 by our protocol.*

Proof. Suppose that Alice(Bob) receives $a(b)$ and sends $c_A(c_B)$. The probability that Alice and Bob obtain basis states $|j\rangle \otimes |j\rangle$ after measurement is

$$\left| \langle j| \otimes \langle j| \left(\text{QFT}_N \otimes \text{QFT}_N^{-1} \right) (P_{a_i} \otimes P_{b_i}) |\Psi_{AB}\rangle \right|^2 = \left| \left(\frac{1}{\sqrt{N}} \right)^3 \sum_{i=0}^{N-1} (-1)^{a_i \oplus b_i} \right|^2.$$

In case of $a = b$, it holds that $a_i \oplus b_i = 0$ for any i . Thus the probability of $c_A = c_B$ is

$$\Pr[c_A = c_B] = 1.$$

On the other case of $a \neq b$, it holds that $d_H(a, b) = \frac{N}{2}$, because of the definition of the Hadamard graph. It means that

$$\Pr[c_A = c_B] = 0.$$

■

By combining Frankl's result [4] and Theorem 3.1, there is a gap between the shared entanglement setting and otherwise for G_{12} , and for the smaller subgraph H mentioned in the previous subsection. We have obtained the following result.

Theorem 3.2 *The smallest Hadamard graph G_N such that the graph colouring game is a pseudo-telepathy game with $c = N$ is G_{12} . Any of its induced subgraphs with 1609 vertices also has this property.*

Godsil and Newman have proved that a Hadamard graph G_N has chromatic number strictly larger than N whenever $N = 4m > 8$ [8]. Then next result holds.

Theorem 3.3 *The graph colouring game for Hadamard graph G_{4m} is a pseudo-telepathy game with $c = 4m$ for all $m \geq 3$.*

The final statement of this theorem uses the result of Frankl and Rodl [5] mentioned in Section 1.

4 Concluding remarks

In this paper, we have dealt with two party case for the quantum colouring game. It may be interesting to investigate the multi-party case for quantum colouring game.

The chromatic number $\chi(G)$ of a graph G is equal to the minimum number of colours such that Alice and Bob win the graph colouring game for G with probability 1 without shared entanglement. Patrick Hayden [private communication] suggested we define the quantum chromatic number $\chi_Q(G)$ as the minimum number of colours such that Alice and Bob win the graph colouring game for G , using shared entanglement, with probability 1. It is easy to see that $\chi_Q(G) \leq \chi(G)$, and the pseudo-telepathy graph colouring game is concerned with graphs with $\chi_Q(G) < \chi(G)$. Characterizing such graphs G would be interesting from both the standpoint of quantum communication and of combinatorics. What is the smallest such graph?

For Hadamard graphs G_{4pq} , there is an exponential gap between the chromatic number and the quantum chromatic number. What is the largest such gap as a function of the number of vertices of G ?

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